

FRACTIONAL DIFFUSION: RECOVERING THE DISTRIBUTED FRACTIONAL DERIVATIVE FROM OVERPOSED DATA

WILLIAM RUNDELL AND ZHIDONG ZHANG

ABSTRACT. There has been considerable recent study in “subdiffusion” models that replace the standard parabolic equation model by a one with a fractional derivative in the time variable. There are many ways to look at this newer approach and one such is to realize that the order of the fractional derivative is related to the time scales of the underlying diffusion process. This raises the question of what order α of derivative should be taken and if a single value actually suffices. This has led to models that combine a finite number of these derivatives each with a different fractional exponent α_k and different weighting value c_k to better model a greater possible range of time scales. Ultimately, one wants to look at a situation that combines derivatives in a continuous way – the so-called distributional model with parameter $\mu(\alpha)$.

However all of this begs the question of how one determines this “order” of differentiation. Recovering a single fractional value has been an active part of the process from the beginning of fractional diffusion modeling and if this is the only unknown then the markers left by the fractional order derivative are relatively straightforward to determine. In the case of a finite combination of derivatives this becomes much more complex due to the more limited analytic tools available for such equations, but recent progress in this direction has been made, [16, 14]. This paper considers the full distributional model where the order is viewed as a function $\mu(\alpha)$ on the interval $(0, 1]$. We show existence, uniqueness and regularity for an initial-boundary value problem including an important representation theorem in the case of a single spatial variable. This is then used in the inverse problem of recovering the distributional coefficient $\mu(\alpha)$ from a time trace of the solution and a uniqueness result is proven.

Keywords: distributed-order fractional diffusion, uniqueness, inverse problem.

AMS subject classifications: 35R30, 26A33, 60C22, 34A08.

1. INTRODUCTION

Classical Brownian motion as formulated in Einstein’s 1905 paper [5] can be viewed as a random walk in which the dynamics are governed by an uncorrelated, Markovian, Gaussian stochastic process. The key assumption is that a change in the direction of motion of a particle is random and that the mean-squared displacement over many changes is proportional to time $\langle x^2 \rangle = Ct$. This easily leads to the derivation of the underlying differential equation being the heat equation.

In fact we can generalize this situation to the case of a continuous time random walk (CTRW) where the length of a given jump, as well as the waiting time elapsing between two successive jumps follow a given probability density function. In one spatial dimension, the picture is as follows: a walker moves along the x -axis, starting at a position x_0 at time $t_0 = 0$. At time t_1 , the walker jumps to x_1 , then at time t_2 jumps to x_2 , and so on. We assume that the temporal and spatial increments $\Delta t_n = t_n - t_{n-1}$, $\Delta x_n = x_n - x_{n-1}$ are independent, identically distributed random

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, USA
E-mail address: rundell@math.tamu.edu zhidong@math.tamu.edu.

variables, following probability density functions $\psi(t)$ and $\lambda(x)$, respectively, which is known as the waiting time distribution and jump length distribution, respectively. Namely, the probability of Δt_n lying in any interval $[a, b] \subset (0, \infty)$ is $P(a < \Delta t_n < b) = \int_a^b \psi(t) dt$ and the probability of Δx_n lying in any interval $[a, b] \subset \mathbb{R}$ is $P(a < \Delta x_n < b) = \int_a^b \lambda(x) dx$. For given ψ and λ , the position x of the walker can be regarded as a step function of t .

It is easily shown using the Central Limit Theorem that provided the first moment, or characteristic waiting time T , defined by $T = \mu_1(\psi) = \int_0^\infty t\psi(t) dt$ and the second moment, or jump length variance Σ , $\mu_2(\lambda) = \int_{-\infty}^\infty x^2\lambda(t) dt$ are finite, then the long-time limit again corresponds to Brownian motion,

On the other hand, when the random walk involves correlations, non-Gaussian statistics or a non-Markovian process (for example, due to “memory” effects) the diffusion equation will fail to describe the macroscopic limit. For example, if we retain the assumption that Σ is finite but relax the condition on a finite characteristic waiting time so that for large t $\psi(t)A/t^{1+\alpha}$ as $t \rightarrow \infty$ where $0 < \alpha \leq 1$, then we get very different results. Such probability density functions are often referred to as a “heavy-tailed.” If in fact we take

$$(1.1) \quad \psi(t) = \frac{A_\alpha}{B_\alpha + t^{1+\alpha}}$$

then again it can be shown, [19, 11], that the effect is to modify the Einstein formulation $\langle x^2 \rangle = Ct$ to $\langle x^2 \rangle = Ct^\alpha$.

This above leads to a *subdiffusive* process and, importantly provides a tractable model where the partial differential equation is replaced by one with a fractional derivative in time of order α . Such objects have been a steady source of investigation over the last almost 200 years beginning in the 1820s with the work of Abel and continuing first by Liouville then by Riemann.

The fractional derivative operator can take several forms, the most usual being either the Riemann-Liouville ${}^R D_0^\alpha$ based on Abel’s original singular integral operator, or the Djrbashian-Caputo ${}^C D_0^\alpha$ version, [4], which reverses the order of the Riemann-Liouville formulation

$$(1.2) \quad \begin{aligned} {}^R D_0^\alpha u &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{\alpha+1-n} u(t) dt, \\ {}^C D_0^\alpha u &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{\alpha+1-n} u^{(n)}(t) dt. \end{aligned}$$

The Džrbašjan-Caputo derivative tends to be more favored by practitioners since it allows the specification of initial conditions in the usual way. Nonetheless, the Riemann-Liouville derivative enjoys certain analytic advantages, including being defined for a wider class of functions and possessing a semigroup property.

Thus the fractional-anomalous diffusion model gives rise to the fractional differential equation

$$(1.3) \quad \partial_t^\alpha u - \mathcal{L}u = f(x, t), \quad x \in \Omega, t \in (0, T)$$

where \mathcal{L} is a uniformly elliptic differential operator on an open domain $\Omega \subset \mathbb{R}^d$ and ∂_t^α is one of the above fractional derivatives. The governing function for the fractional derivative becomes the Mittag-Leffler function $E_{\alpha, \beta}(z)$ which generalizes the exponential function that forms the key component for the fundamental solution in the classical case when $\alpha = \beta = 1$.

$$(1.4) \quad E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

For the typical examples described here we have $0 < \alpha \leq 1$ and β a positive real number although further generalization is certainly possible. See, for example, [7].

During the past two decades, differential equations involving fractional-order derivatives have received increasing attention in applied disciplines. Such models are known to capture more faithfully the dynamics of anomalous diffusion processes in amorphous materials, e.g., viscoelastic materials, porous media, diffusion on domains with fractal geometry and option pricing models. These models also describe certain diffusion processes more accurately than Gaussian-based Brownian motion and have particular relevance to materials exhibiting memory effects. As a consequence, we can obtain fundamentally different physics. There has been significant progress on both mathematical methods and numerical algorithm design and, more recently, attention has been paid to inverse problems. This has shed considerable light on the new physics appearing, [10, 23]

Of course, such a specific form for $\psi(t)$ as given by 1.1 is rather restrictive as it assumes a quite specific scaling factor between space and time distributions and there is no reason to expect nature is so kind to only require a single value for α .

One approach around this is to take a finite sum of such terms each corresponding to a different value of α . This leads to a model where the time derivative is replaced by a finite sum of fractional derivatives of orders α_j and by analogy leads to the law $\langle x^2 \rangle = g(t, \alpha)$ where g is a finite sum of fractional powers. This formulation replaces the single value fractional derivative by a finite sum $\sum_1^m q_j \partial_t^{\alpha_j} u$ where a linear combination of m fractional powers has been taken. Physically this represents a fractional diffusion model that assumes diffusion takes place in a medium in which there is no single scaling exponent; for example, a medium in which there are memory effects over multiple time scales.

This seemingly simple device leads to considerable complications. For one, we have to use the so-called multi-index Mittag-Leffler function $E_{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m}(z)$ in place of the two parameter $E_{\alpha, \beta}(z)$ and this adds complexity not only notationally but in proving required regularity results for the basic forwards problem of knowing $\Omega, \mathcal{L}, f, u_0$ and recovering $u(x, t)$, see [16, 14] and the references within.

It is also possible to generalize beyond the finite sum by taking the so-called distributed fractional derivative,

$$(1.5) \quad \partial_t^{(\mu)} u(t) = \int_0^1 \mu(\alpha) \partial_t^\alpha u(t) d\alpha.$$

Thus the finite sum derivative can be obtained by taking $\mu(\alpha) = \sum_{j=1}^m q_j \delta(\alpha - \alpha_j)$. See [20, 12, 18, 17, 15], for several studies incorporating this extension. This in turn allows a more general function probability density distribution function ψ in 1.1 and hence a more general value for $g(t, \alpha)$.

The purpose of this paper is to analyze this distributed model extension to equation (1.3) and the paper is organized as follows. First, we demonstrate existence, uniqueness and regularity results for the solution of the distributed fractional derivative model on a cylindrical region in space-time $\Omega \times [0, T]$ where Ω is a bounded, open set in \mathbb{R}^d . Second, in the case of one spatial variable, $d = 1$, we set up representation theorems for the solution analogous to that for the heat equation itself, [2], and extended to the case of a single fractional derivative in [21].

Section 2 looks at the assumptions to be made on the various terms in (1.5) and utilizes these to show existence, uniqueness and regularity results for the direct problem; namely, to be given $\Omega, \mathcal{L}, f, u_0$ and the function $\mu = \mu(\alpha)$, then to solve (1.5) for $u(x, t)$.

Section 4 will derive several representation theorems for this solution and these will be used in the final section to formulate and prove a uniqueness result for the associated inverse problem to be discussed below.

However, there is the obvious question for all of these models: what is the value of α ? Needless to say there has been much work done on this; experiments have been set up to collect additional information that allows a best fit for α in a given setting. One of the earliest works here is from 1975, [22] and in part was based on the Montroll-Weiss random walk model [19]. See also [8]. Mathematically the recovery in models with a single value for α turns out to be relatively straightforward provided we are able to choose the type of data being measured. This would be chosen to allow us to rely on the known asymptotic behavior of the Mittag-Leffler function for both small and large arguments. An exception here is when we also have to determine α as well as an unknown coefficient in which case the combination problem can be decidedly much more complex. See, for example, [3, 13, 21]. Amongst the first papers in this direction with a rigorous existence and uniqueness analysis is [9].

The multi-term case, although similar in concept, is quite nontrivial but has been shown in [16, 14]. In these papers the authors were able to prove an important uniqueness theorem: if given the additional data consisting of the value of the normal derivative $\frac{\partial u}{\partial \nu}$ at a fixed point $x_0 \in \partial\Omega$ for all t then the sequence pair $\{q_j, \alpha_j\}_{j=1}^m$ can be uniquely recovered.

The main result of the current paper in this direction is in Section 5 where we show that the uniqueness results of [16, 14] can be extended to recover a suitably defined exponent function $\mu(\alpha)$.

2. PRELIMINARY MATERIAL

Let Ω be an open bounded domain in \mathbb{R}^d with a smooth (C^2 will be more than sufficient) boundary $\partial\Omega$ and let $T > 0$ be a fixed constant.

\mathcal{L} is a strongly elliptic, self-adjoint operator with smooth coefficients defined on Ω ,

$$\mathcal{L}u = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u$$

where $a_{ij}(x) \in C^1(\overline{\Omega})$, $c(x) \in C(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ and $\sum_{i,j=1}^d a_{ij} \xi_i \xi_j \geq \delta \sum_{i=1}^d \xi_i^2$ for some $\delta > 0$, all $x \in \overline{\Omega}$ and all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$.

To avoid unnecessary complications for the main theme we will make the assumption of homogeneous Dirichlet boundary conditions on $\partial\Omega$ so that the natural domain for \mathcal{L} is $H^2(\Omega) \cap H_0^1(\Omega)$. Then $-\mathcal{L}$ has a complete, orthonormal system of eigenfunctions $\{\psi_n\}_{n=1}^\infty$ in $L^2(\Omega)$ with $\psi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and with corresponding eigenvalues $\{\lambda_n\}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.

The nonhomogeneous term will be taken to satisfy $f(x, t) \in C(0, T; H^2(\Omega))$. This can be weakened to assume only L^p regularity in time, but as shown in [14] this requires more delicate analysis. The initial value $u_0(x) \in H^2(\Omega)$. We will use $\langle \cdot, \cdot \rangle$ to denote the inner product in $L^2(\Omega)$.

Throughout this paper we will, by following [12], make the assumptions on the distributed derivative parameter μ .

Assumption 2.1.

$$\mu \in C^1[0, 1], \mu(\alpha) \geq 0, \mu(1) \neq 0.$$

Remark 2.1. From these conditions it follows that there exists a constant $C_\mu > 0$ and an interval $(\beta_0, \beta) \subset (0, 1)$ such that $\mu(\alpha) \geq C_\mu$ on (β_0, β) . This will be needed in our proof of the representation theorem in Section 4.

We will use the Djrbashian-Caputo version for $D^{(\mu)}$: $D^{(\mu)}u = \int_0^1 \mu(\alpha) \partial_t^\alpha u d\alpha$ with $\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} u(x, \tau) d\tau$ and so

$$(2.1) \quad D^{(\mu)}u = \int_0^t \left[\int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)} (t-\tau)^{-\alpha} d\alpha \right] \frac{d}{d\tau} u(x, \tau) d\tau := \int_0^t \eta(t-\tau) \frac{d}{d\tau} u(x, \tau) d\tau,$$

where

$$(2.2) \quad \eta(s) = \int_0^1 \frac{\mu(\alpha)}{\Gamma(1-\alpha)} s^{-\alpha} d\alpha.$$

Thus our distributed differential equation (DDE) model in this paper will be

$$(2.3) \quad \begin{aligned} D^{(\mu)}u(x, t) - \mathcal{L}u(x, t) &= f(x, t), & x \in \Omega, \quad t \in (0, T); \\ u(x, t) &= 0, & x \in \partial\Omega, \quad t \in (0, T); \\ u(x, 0) &= u_0(x), & x \in \Omega. \end{aligned}$$

2.1. A Distributional ODE. Our first task is to analyze the ordinary distributed fractional order equation

$$(2.4) \quad D^{(\mu)}v(t) = -\lambda v(t), \quad v(0) = 1, \quad t \in (0, T)$$

and to show there exists a unique solution. We will need some preliminary analysis to determine the integral operator that serves as the inverse for $D^{(\mu)}$ in analogy with the Riemann-Liouville derivative being inverted by the Abel operator. If we now take the Laplace transform of η in (2.2) then we have

$$(2.5) \quad (\mathcal{L}\eta)(z) = \frac{\Phi(z)}{z}, \quad \text{where } \Phi(z) = \int_0^1 \mu(\alpha) z^\alpha d\alpha.$$

The next lemma introduces an operator $I^{(\mu)}$ to analyze the distributed ODE (2.4).

Lemma 2.1. *Define the operator $I^{(\mu)}$ as*

$$I^{(\mu)}\phi(t) = \int_0^t \kappa(t-s)\phi(s)ds, \quad \text{where } \kappa(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{zt}}{\Phi(z)} dz.$$

Then the following conclusions hold:

- (1) $D^{(\mu)}I^{(\mu)}\phi(t) = \phi(t)$, $I^{(\mu)}D^{(\mu)}\phi(t) = \phi(t) - \phi(0)$ for $\phi \in C^1(0, T)$;
- (2) $\kappa(t) \in C^\infty(0, \infty)$ and

$$(2.6) \quad \kappa(t) = |\kappa(t)| \leq C \ln \frac{1}{t} \quad \text{for sufficiently small } t > 0.$$

Proof. This is [12, Proposition 3.2]. We remark that the result in this paper include further estimates on κ that require additional regularity on μ . However, for the bound (2.6) only C^1 regularity on μ is needed. \square

Remark 2.2. In [12, Proposition 3.2], if the condition either $\mu(0) \neq 0$ or $\mu(\alpha) \sim a\alpha^v$, $a > 0$, $v > 0$ is added, then κ is completely monotone. This property is not explicitly used in this paper, however as we remark after the uniqueness result, this condition on κ could be a useful basis for a reconstruction algorithm.

With $I^{(\mu)}$, we have the following results.

Lemma 2.2. *For each $\lambda > 0$ there exists a unique $u(t)$ which satisfies (2.4).*

Proof. Lemma 2.1 implies that (2.4) is equivalent to

$$u(t) = -\lambda I^{(\mu)} u(t) + 1 =: A_1 u.$$

Now the asymptotic and smoothness results of $\kappa(t)$ in Lemma 2.1 give $\kappa \in L^1(0, T)$, that is, there exists $t_1 \in (0, T)$ such that

$$\|\kappa\|_{L^1(0, t_1)} < \frac{1}{\lambda}.$$

Hence, given $\phi_1, \phi_2 \in L^1(0, t_1)$,

$$\begin{aligned} \|A_1(\phi_1) - A_1(\phi_2)\|_{L^1(0, t_1)} &\leq \lambda \int_0^{t_1} \int_0^t |\kappa(t-s)| \cdot |\phi_1(s) - \phi_2(s)| ds dt \\ &= \lambda \int_0^{t_1} |\phi_1(s) - \phi_2(s)| \int_s^{t_1} |\kappa(t-s)| dt ds \\ &\leq \lambda \int_0^{t_1} |\phi_1(s) - \phi_2(s)| \cdot \|\kappa\|_{L^1(0, t_1)} ds \\ &= \lambda \|\kappa\|_{L^1(0, t_1)} \cdot \|\phi_1 - \phi_2\|_{L^1(0, t_1)}. \end{aligned}$$

From the fact that $0 < \lambda \|\kappa\|_{L^1(0, t_1)} < 1$, A_1 is a contraction map on $L^1(0, t_1)$ and so by the Banach fixed point theorem, there exists a unique $u_1(t) \in L^1(0, t_1)$ that satisfies $u_1 = A_1 u_1$.

For each $t \in (t_1, 2t_1)$, we have

$$u(t) = 1 - \lambda I^{(\mu)} u(t) = 1 - \lambda \int_{t_1}^t \kappa(t-s) u(s) ds - \lambda \int_0^{t_1} \kappa(t-s) u(s) ds.$$

Since $u = u_1$ on $(0, t_1)$ which is now known, then

$$u(t) = -\lambda \int_{t_1}^t \kappa(t-s) u(s) ds + 1 - \lambda \int_0^{t_1} \kappa(t-s) u_1(s) ds := A_2 u$$

for each $t \in (t_1, 2t_1)$. Given $\phi_1, \phi_2 \in L^1(t_1, 2t_1)$, it holds

$$\begin{aligned} \|A_2(\phi_1) - A_2(\phi_2)\|_{L^1(t_1, 2t_1)} &\leq \lambda \int_{t_1}^{2t_1} \int_{t_1}^t |\kappa(t-s)| \cdot |\phi_1(s) - \phi_2(s)| ds dt \\ &= \lambda \int_{t_1}^{2t_1} |\phi_1(s) - \phi_2(s)| \int_s^{2t_1} |\kappa(t-s)| dt ds \\ &\leq \lambda \int_{t_1}^{2t_1} |\phi_1(s) - \phi_2(s)| \cdot \|\kappa\|_{L^1(0, t_1)} ds \\ &= \lambda \|\kappa\|_{L^1(0, t_1)} \cdot \|\phi_1 - \phi_2\|_{L^1(t_1, 2t_1)}. \end{aligned}$$

Hence, A_2 is also a contraction map on $L^1(t_1, 2t_1)$, which yields and shows that there exists a unique $u_2(t) \in L^1(t_1, 2t_1)$ such that $u_2 = A_2 u_2$.

Repeating this argument yields that there exists a unique solution $u \in L^1(0, T)$ of the distributed ODE (2.4), which completes the proof. \square

Lemma 2.3. $u(t) \in C^\infty(0, T)$ is completely monotone, which gives $0 \leq u(t) \leq 1$ on $[0, T]$.

Proof. This lemma is a special case of [12, Theorem 2.3]. \square

3. EXISTENCE, UNIQUENESS AND REGULARITY

3.1. Existence and uniqueness of weak solution for DDE (2.3) . We state the definition of the weak solution as

Definition 3.1. $u(x, t)$ is a weak solution to DDE (2.3) in $L^2(\Omega)$ if $u(\cdot, t) \in H_0^1(\Omega)$ for $t \in (0, T)$ and for any $\psi(x) \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\begin{aligned} \langle D^{(\mu)} u(x, t), \psi(x) \rangle - \langle \mathcal{L} u(x, t; a), \psi(x) \rangle &= \langle f(x, t), \psi(x) \rangle, \quad t \in (0, T); \\ \langle u(x, 0), \psi(x) \rangle &= \langle u_0(x), \psi(x) \rangle. \end{aligned}$$

Then Lemma 2.2 gives the following corollary.

Corollary 3.1. There exists a unique weak solution $u^*(x, t)$ of DDE (2.3) and the representation of $u^*(x, t)$ is

$$(3.1) \quad \begin{aligned} u^*(x, t) &= \sum_{n=1}^{\infty} \left[\langle u_0, \psi_n \rangle u_n(t) + \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \right. \\ &\quad \left. + \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \right] \psi_n(x), \end{aligned}$$

where $u_n(t)$ is the unique solution of the distributed ODE (2.4) with $\lambda = \lambda_n$.

Proof. Completeness of $\{\psi_n(x) : n \in \mathbb{N}^+\}$ in $L^2(\Omega)$ and direct calculation show that the representation (3.1) is a weak solution of DDE (2.3); while the uniqueness of u^* follows from Lemma 2.2. \square

3.2. Regularity. The next two lemmas concern the regularity of u^* and $D^{(\mu)} u^*$.

Lemma 3.1.

$$\|u^*(x, t)\|_{C([0, T]; H^2(\Omega))} \leq C(\|u_0\|_{H^2(\Omega)} + \|f(\cdot, 0)\|_{H^2(\Omega)} + T^{1/2} \|f\|_{H^1([0, T]; H^2(\Omega))})$$

where $C > 0$ depends on μ , \mathcal{L} and Ω , and $\|f\|_{H^1([0, T]; H^2(\Omega))} = \left\| \frac{\partial f}{\partial t} \right\|_{L^2([0, T]; H^2(\Omega))}$.

Proof. Fix $t \in (0, T)$,

$$\begin{aligned} \|u^*(x, t)\|_{H^2(\Omega)} &\leq \left\| \sum_{n=1}^{\infty} \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right\|_{H^2(\Omega)} &&:= I_1 \\ &\quad + \left\| \sum_{n=1}^{\infty} \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \psi_n(x) \right\|_{H^2(\Omega)} &&:= I_2 \\ &\quad + \left\| \sum_{n=1}^{\infty} \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \psi_n(x) \right\|_{H^2(\Omega)} &&:= I_3. \end{aligned}$$

We estimate each of I_1 , I_2 , and I_3 in turn using Lemmas 2.1 and 2.3 where in each case $C > 0$ is a generic constant that depends only on μ , \mathcal{L} and Ω .

$$\begin{aligned} I_1^2 &= \left\| \sum_{n=1}^{\infty} \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right\|_{H^2(\Omega)}^2 \leq C \left\| \mathcal{L} \left(\sum_{n=1}^{\infty} \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right) \right\|_{L^2(\Omega)}^2 \\ &= C \left\| \sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right\|_{L^2(\Omega)}^2 = C \sum_{n=1}^{\infty} \lambda_n^2 \langle u_0, \psi_n \rangle^2 u_n^2(t) \\ &\leq C \sum_{n=1}^{\infty} \lambda_n^2 \langle u_0, \psi_n \rangle^2 = C \|\mathcal{L} u_0\|_{L^2(\Omega)}^2 \leq C \|u_0\|_{H^2(\Omega)}^2. \end{aligned}$$

$$\begin{aligned}
I_2^2 &= \left\| \sum_{n=1}^{\infty} \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \psi_n(x) \right\|_{H^2(\Omega)}^2 \leq C \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot, 0), \psi_n \rangle^2 (I^{(\mu)} u_n(t))^2 \\
&\leq C \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot, 0), \psi_n \rangle^2 \left(\int_0^t |\kappa(\tau)| \cdot |u_n(t - \tau)| d\tau \right)^2 \\
&\leq C \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot, 0), \psi_n \rangle^2 \left(\int_0^t |\kappa(\tau)| d\tau \right)^2 \\
&\leq C \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot, 0), \psi_n \rangle^2 \|\kappa\|_{L^1(0,T)}^2 \leq C \|\kappa\|_{L^1(0,T)}^2 \|f(\cdot, 0)\|_{H^2(\Omega)}^2. \\
I_3^2 &= \left\| \sum_{n=1}^{\infty} \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \psi_n(x) \right\|_{H^2(\Omega)}^2 \\
&\leq C \sum_{n=1}^{\infty} \left[\int_0^t \lambda_n \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \right]^2 \\
&\leq C \sum_{n=1}^{\infty} \left[\int_0^t \lambda_n \left| \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle \right| \cdot |I^{(\mu)} u_n(t - \tau)| d\tau \right]^2 \\
&\leq C \|\kappa\|_{L^1(0,T)}^2 \sum_{n=1}^{\infty} \int_0^t \lambda_n^2 \left| \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle \right|^2 d\tau \cdot \int_0^t 1^2 d\tau \\
&\leq CT \|\kappa\|_{L^1(0,T)}^2 \int_0^T \sum_{n=1}^{\infty} \lambda_n^2 \left| \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle \right|^2 d\tau \leq CT \|\kappa\|_{L^1(0,T)}^2 \int_0^T \left\| \frac{\partial}{\partial t} f(\cdot, \tau) \right\|_{H^2(\Omega)}^2 d\tau \\
&= CT \|\kappa\|_{L^1(0,T)}^2 \|f\|_{H^1([0,T];H^2(\Omega))}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|u^*(x, t)\|_{C([0,T];H^2(\Omega))} &\leq C \|u_0\|_{H^2(\Omega)} + C \|\kappa\|_{L^1(0,T)} \|f(\cdot, 0)\|_{H^2(\Omega)} \\
&\quad + CT^{1/2} \|\kappa\|_{L^1(0,T)} \|f\|_{H^1([0,T];H^2(\Omega))} \\
&\leq C (\|u_0\|_{H^2(\Omega)} + \|f(\cdot, 0)\|_{H^2(\Omega)} + T^{1/2} \|f\|_{H^1([0,T];H^2(\Omega))}).
\end{aligned}$$

Due to the fact that κ is determined by μ , the constant C above only depends on μ , \mathcal{L} and Ω . \square

Lemma 3.2.

$$\|D^{(\mu)} u^*\|_{C([0,T];L^2(\Omega))} \leq C \left(\|u_0\|_{H^2(\Omega)} + T^{1/2} \|f\|_{H^1([0,T];H^2(\Omega))} + \|f\|_{C([0,T];H^2(\Omega))} \right),$$

where $C > 0$ only depends on μ , \mathcal{L} and Ω .

Proof. For each $t \in (0, T)$,

$$\begin{aligned}
D^{(\mu)} u^*(x, t) &= - \sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) - \sum_{n=1}^{\infty} \lambda_n \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \psi_n(x) \\
&\quad - \sum_{n=1}^{\infty} \lambda_n \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \psi_n(x) + f(x, t),
\end{aligned}$$

which implies

$$\begin{aligned}
\|D^{(\mu)} u^*\|_{L^2(\Omega)} &\leq \left\| \sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right\|_{L^2(\Omega)} + \left\| \sum_{n=1}^{\infty} \lambda_n \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \psi_n(x) \right\|_{L^2(\Omega)} \\
&\quad + \left\| \sum_{n=1}^{\infty} \lambda_n \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \psi_n(x) \right\|_{L^2(\Omega)} + \|f(\cdot, t)\|_{L^2(\Omega)}.
\end{aligned}$$

Combining the estimates for I_1 , I_2 and I_3 we obtain

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \lambda_n \langle u_0, \psi_n \rangle u_n(t) \psi_n(x) \right\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \langle u_0, \psi_n \rangle^2 u_n^2(t) \leq C \|u_0\|_{H^2(\Omega)}^2, \\ \left\| \sum_{n=1}^{\infty} \lambda_n \langle f(\cdot, 0), \psi_n \rangle I^{(\mu)} u_n(t) \psi_n(x) \right\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \langle f(\cdot, 0), \psi_n \rangle^2 (I^{(\mu)} u_n(t))^2 \\ &\leq C \|\kappa\|_{L^1(0,T)}^2 \|f(\cdot, 0)\|_{H^2(\Omega)}^2 \\ &\leq C \|\kappa\|_{L^1(0,T)}^2 \|f\|_{C([0,T];H^2(\Omega))}^2 \end{aligned}$$

and

$$\begin{aligned} &\left\| \sum_{n=1}^{\infty} \lambda_n \int_0^t \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \psi_n(x) \right\|_{L^2(\Omega)}^2 \\ &= \sum_{n=1}^{\infty} \left[\int_0^t \lambda_n \left\langle \frac{\partial}{\partial t} f(\cdot, \tau), \psi_n \right\rangle I^{(\mu)} u_n(t - \tau) d\tau \right]^2 \leq CT \|\kappa\|_{L^1(0,T)}^2 \|f\|_{H^1([0,T];H^2(\Omega))}^2. \end{aligned}$$

Therefore,

$$\|D^{(\mu)} u^*\|_{C([0,T];L^2(\Omega))} \leq C \left(\|u_0\|_{H^2(\Omega)} + T^{1/2} \|f\|_{H^1([0,T];H^2(\Omega))} + \|f\|_{C([0,T];H^2(\Omega))} \right),$$

where C is dependent only on μ , \mathcal{L} and Ω . \square

The main theorem of this section follows from Corollary 3.1, Lemmas 3.1 and 3.2.

Theorem 3.2 (Main theorem for the direct problem). *There exists a unique weak solution $u^*(x, t)$ in $L^2(\Omega)$ of the DDE (2.3) with the representation (3.1) and the following regularity estimate*

$$\begin{aligned} \|u^*\|_{C([0,T];H^2(\Omega))} + \|D^{(\mu)} u^*\|_{C([0,T];L^2(\Omega))} \\ \leq C \left(\|u_0\|_{H^2(\Omega)} + T^{1/2} \|f\|_{H^1([0,T];H^2(\Omega))} + \|f\|_{C([0,T];H^2(\Omega))} \right), \end{aligned}$$

where $C > 0$ depends only on μ , \mathcal{L} and Ω .

4. REPRESENTATION OF THE DDE SOLUTION FOR ONE SPATIAL VARIABLE

In this section, we will establish a representation result for the special case $\Omega = (0, 1)$, $\mathcal{L}u = u_{xx}$ in (2.3)

$$(4.1) \quad \begin{cases} D^{(\mu)} u - u_{xx} = f(x, t), & 0 < x < 1, \quad 0 < t < \infty; \\ u(x, 0) = u_0(x), & 0 < x < 1; \\ u(0, t) = g_0(t), & 0 \leq t < \infty; \\ u(1, t) = g_1(t), & 0 \leq t < \infty, \end{cases}$$

where $g_0, g_1 \in L^2(0, \infty)$ and $f(x, \cdot) \in L^1(0, \infty)$ for each $x \in (0, 1)$.

We can obtain the fundamental solution by Laplace and Fourier transforms. First, we extend the finite domain to an infinite one and impose a homogeneous right-hand side, i.e. we consider the following model

$$\begin{cases} D^{(\mu)} u - u_{xx} = 0, & -\infty < x < \infty, \quad 0 < t < \infty; \\ u(x, 0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

Next we take the Fourier transform \mathcal{F} with respect to x and denote $(\mathcal{F}u)(\xi, t)$ by $\tilde{u}(\xi, t)$,

$$D^{(\mu)} \tilde{u}(\xi, t) + \xi^2 \tilde{u}(\xi, t) = 0.$$

Then by taking the Laplace transform \mathbb{L} with respect to t and denote $(\mathbb{L}\hat{u})(\xi, z)$ by $\hat{u}(\xi, z)$, we obtain

$$\int_0^1 \mu(\alpha) \left(z^\alpha \hat{u}(\xi, z) - z^{\alpha-1} \tilde{u}_0(\xi) \right) d\alpha + \xi^2 \hat{u}(\xi, z) = 0,$$

that is,

$$\hat{u}(\xi, z) = \frac{\Phi(z)/z}{\Phi(z) + \xi^2} \tilde{u}_0(\xi),$$

where $\Phi(z)$ comes from (2.5).

Then we have

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \circ \mathbb{L}^{-1}(\hat{u}(\xi, z)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} \frac{\Phi(z)/z}{\Phi(z) + \xi^2} \tilde{u}_0(\xi) dz d\xi \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z) + \xi^2} \tilde{u}_0(\xi) d\xi dz \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} (\mathcal{F}^{-1}(\frac{\Phi(z)/z}{\Phi(z) + \xi^2}) * u_0)(x) dz, \end{aligned}$$

where the integral above is the usual Bromwich path, that is, a line in the complex plane parallel to the imaginary axis $z = \gamma + it$, $-\infty < t < \infty$, see [25]. The last equality follows from the Fourier transform formula on convolutions and γ can be an arbitrary positive number due to the fact that $z = 0$ is a singular point of the function $\frac{\Phi(z)/z}{\Phi(z) + \xi^2}$. Throughout the remainder of this paper we will use γ to denote a strictly positive constant which is larger than $e^{1/\beta}$. The number $e^{1/\beta}$ will be seen in the proof of Lemma 4.3. We shall assume the angle of variation z for the Laplace transforms is from $-\pi$ to π , that is $z \in \Lambda := \{z \in \mathbb{C} : \arg(z) \in (-\pi, \pi]\}$.

For $\Phi(z)$, we have the following result which will be central to the rest of the paper. It can be shown by using the Cauchy-Riemann equations in polar form.

Lemma 4.1. $\Phi(z)$ is analytic on $\mathbb{C} \setminus \{0\}$.

In the next two lemmas, we obtain important properties of $\Phi(z)$.

Lemma 4.2. $\operatorname{Re}(\Phi^{1/2}(z)) \geq \frac{\sqrt{2}}{2} |\Phi^{1/2}(z)|$, $\operatorname{Re} z = \gamma > 0$.

Proof. $\gamma > 0$ implies that $\operatorname{Re} z > 0$, i.e. $\arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which together with $0 < \alpha < 1$ and $\mu(\alpha) \geq 0$ yields $\operatorname{Re} \Phi(z) \geq 0$, i.e. $\arg(\Phi(z)) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This gives $\arg(\Phi^{1/2}(z)) \in (-\frac{\pi}{4}, \frac{\pi}{4})$. Hence,

$$\operatorname{Re}(\Phi^{1/2}(z)) = \cos(\arg(\Phi^{1/2}(z))) |\Phi^{1/2}(z)| \geq \frac{\sqrt{2}}{2} |\Phi^{1/2}(z)|,$$

which completes the proof. \square

Lemma 4.3.

$$C_{\mu, \beta} \frac{\gamma^\beta - \gamma^{\beta_0}}{\ln \gamma} \leq C_{\mu, \beta} \frac{|z|^\beta - |z|^{\beta_0}}{\ln |z|} \leq |\Phi(z)| \leq C \frac{|z| - 1}{\ln |z|},$$

for z such that $\operatorname{Re} z = \gamma > e^{1/\beta} > 0$.

Proof. For the right-hand side of the inequality, $\mu(\alpha) \in C^1[0, 1]$ obviously implies that there exists a $C > 0$ such that $|\mu(\alpha)| \leq C$ on $[0, 1]$. Hence,

$$|\Phi(z)| \leq \int_0^1 |\mu(\alpha)| \cdot |z|^\alpha d\alpha \leq C \int_0^1 |z|^\alpha d\alpha = C \frac{|z| - 1}{\ln |z|}.$$

For the left-hand side, write $z = re^{i\theta}$. Since $\operatorname{Re} z = \gamma > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then

$$\begin{aligned} |\Phi(z)| &\geq \operatorname{Re}(\phi(z)) = \int_0^1 \mu(\alpha) r^\alpha \cos(\theta\alpha) d\alpha \\ &\geq C_\mu \int_{\beta_0}^\beta r^\alpha \cos(\theta\alpha) d\alpha \geq C_\mu \cos(\beta\theta) \int_{\beta_0}^\beta r^\alpha d\alpha \\ &\geq C_\mu \cos\left(\frac{\beta\pi}{2}\right) \int_{\beta_0}^\beta |z|^\alpha d\alpha = C_{\mu,\beta} \frac{|z|^\beta - |z|^{\beta_0}}{\ln |z|}. \end{aligned}$$

Recall $|z| \geq \gamma > e^{1/\beta}$, we have $\frac{|z|^\beta - |z|^{\beta_0}}{\ln |z|} \geq \frac{\gamma^\beta - \gamma^{\beta_0}}{\ln \gamma}$ due to the function $\frac{x^\beta - x^{\beta_0}}{\ln x}$ being increasing on the interval $(e^{1/\beta}, +\infty)$. \square

Now we are in a position to calculate the complex integral $\mathcal{F}^{-1}\left(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\right)$.

Lemma 4.4. $\mathcal{F}^{-1}\left(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\right) = \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x|}$.

Proof. From the inverse Fourier transform formula we have

$$\mathcal{F}^{-1}\left(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi.$$

We denote the contour from $-R$ to R by C_0 , the semicircle with radius R in the upper and lower half plane by C_{R+} and C_{R-} , respectively. Also, let C_+ , C_- be the closed contours which consist of C_0, C_{R+} and C_0, C_{R-} respectively.

For the case of $x > 0$, working on the closed contour C_+ , we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi - \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{C_{R+}} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi \\ &= \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi, \end{aligned}$$

where the second limit is 0 as follows from Jordan's Lemma. Since to $0 < \alpha < 1$, $\gamma > 0$, by our assumptions we have $\operatorname{Re}(\Phi(z)) \geq 0$, which in turn leads to $\operatorname{Re}(\Phi^{1/2}(z)) \geq 0$. Then there is only one singular point $\xi = i\Phi^{1/2}(z)$ in C_+ which is contained by the upper half plane. By the residue theorem [25], we have

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_+} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi = \lim_{R \rightarrow \infty} 2\pi i \frac{1}{2\pi} e^{ixi\Phi^{1/2}(z)} \frac{\Phi(z)/z}{2i\Phi^{1/2}(z)} = \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)x}.$$

For the case of $x < 0$, we choose the closed contour C_- . Since $\operatorname{Re}(\Phi^{1/2}(z)) \geq 0$, it follows that $\xi = -i\Phi^{1/2}(z)$ is the unique singular point in C_- . Then a similar calculation gives

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi &= - \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_-} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi + \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{C_{R-}} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi \\ &= - \lim_{R \rightarrow \infty} \frac{1}{2\pi} \oint_{C_-} e^{ix\xi} \frac{\Phi(z)/z}{\Phi(z)+\xi^2} d\xi \\ &= \lim_{R \rightarrow \infty} \frac{\Phi^{1/2}(z)}{2z} e^{\Phi^{1/2}(z)x} = \frac{\Phi^{1/2}(z)}{2z} e^{\Phi^{1/2}(z)x}. \end{aligned}$$

Therefore,

$$\mathcal{F}^{-1}\left(\frac{\Phi(z)/z}{\Phi(z)+\xi^2}\right) = \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x|},$$

which completes the proof. \square

4.1. **The fundamental solution** $G_\mu(x, t)$. With the above lemma, we have

$$\begin{aligned} u(x, t) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} \int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x-y|} u_0(y) dy dz \\ &= \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt-\Phi^{1/2}(z)|x-y|} dz \right] u_0(y) dy. \end{aligned}$$

Then we can define the fundamental solution $G_{(\mu)}(x, t)$ as

$$(4.2) \quad G_{(\mu)}(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt-\Phi^{1/2}(z)|x|} dz.$$

The following three lemmas provide some important properties of $G_{(\mu)}(x, t)$.

Lemma 4.5. *The integral for $G_{(\mu)}(x, t)$ is convergent for each $(x, t) \in (0, \infty) \times (0, \infty)$.*

Proof. Given $(x, t) \in (0, \infty) \times (0, \infty)$, with Lemmas 4.2 and 4.3, we have

$$\begin{aligned} |G_{(\mu)}(x, t)| &\leq \frac{1}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Phi^{1/2}(z)}{z} \right| \cdot |e^{zt}| \cdot |e^{-\Phi^{1/2}(z)|x|}| dz \\ &= \frac{1}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{|\Phi^{1/2}(z)|}{|z|} e^{\gamma t} e^{-\operatorname{Re}(\Phi^{1/2}(z)|x|)} dz \\ &\leq \frac{1}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{|\Phi^{1/2}(z)|}{|z|} e^{\gamma t} e^{-\frac{\sqrt{2}}{2}|x||\Phi^{1/2}(z)|} dz \\ &\leq \frac{C e^{\gamma t}}{4\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} (|z| \ln |z|)^{-1/2} e^{-C_{\mu, \beta}|x|(\frac{|z|^\beta - |z|^{\beta_0}}{\ln |z|})^{1/2}} dz \\ &\leq \frac{C e^{\gamma t}}{4\pi (\ln \gamma)^{1/2}} \int_{\gamma-i\infty}^{\gamma+i\infty} |z|^{-1/2} e^{-C_{\mu, \beta}|x|(\frac{C|z|^\beta}{\ln |z|})^{1/2}} dz < \infty. \end{aligned}$$

□

Lemma 4.6. $G_{(\mu)}(x, t) \in C^\infty((0, \infty) \times (0, \infty))$.

Proof. Fix $(x, t) \in (0, \infty) \times (0, \infty)$. Then for small $|\epsilon_x|, |\epsilon_t|$ we have

$$\begin{aligned} |G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t)| &\leq |G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t + \epsilon_t)| \\ &\quad + |G_{(\mu)}(x, t + \epsilon_t) - G_{(\mu)}(x, t)|. \end{aligned}$$

For $|G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t + \epsilon_t)|$, the following holds

$$\begin{aligned} &|G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t + \epsilon_t)| \\ &\leq \frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}| \cdot |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}| dz. \end{aligned}$$

From the proof of Lemma 4.5, we have

$$\begin{aligned} |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}| &\leq |e^{-\Phi^{1/2}(z)(\frac{x}{2}+\epsilon_x)}| + |e^{-\Phi^{1/2}(z)(x/2)}| \\ &\leq e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(\frac{x}{2}+\epsilon_x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(x/2)} \leq 2, \end{aligned}$$

and

$$\frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{zt+z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2|}| dz < \infty.$$

Hence, after setting $e_1(z, \epsilon_x) = |e^{-\Phi^{1/2}(z)(\frac{x}{2} + \epsilon_x)} - e^{-\Phi^{1/2}(z)(x/2)}|$, we can apply Lebesgue's dominated convergent theorem to deduce that

$$\begin{aligned} & \lim_{\epsilon_x \rightarrow 0} |G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t + \epsilon_t)| \\ & \leq \lim_{\epsilon_x \rightarrow 0} \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{zt + z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2}| \cdot e_1(z, \epsilon_x) \, dz \\ & = \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{zt + z\epsilon_t}| \cdot |e^{-\Phi^{1/2}(z)|x/2}| \cdot \lim_{\epsilon_x \rightarrow 0} e_1(z, \epsilon_x) \, dz = 0. \end{aligned}$$

A similar argument also shows that $\lim_{\epsilon_t \rightarrow 0} |G_{(\mu)}(x, t + \epsilon_t) - G_{(\mu)}(x, t)| = 0$. From this we deduce that $\lim_{\epsilon_x, \epsilon_t \rightarrow 0} |G_{(\mu)}(x + \epsilon_x, t + \epsilon_t) - G_{(\mu)}(x, t)| = 0$, which shows that $G_{(\mu)}(x, t) \in C((0, \infty) \times (0, \infty))$.

Similarly, following from the proof of Lemma 4.5 and the above limiting argument, we obtain

$$G_{(\mu)}(x, t) \in C^m((0, \infty) \times (0, \infty)), \quad n \in \mathbb{N}^+,$$

which leads to $G_{(\mu)}(x, t) \in C^\infty((0, \infty) \times (0, \infty))$ and this completes the proof. \square

Lemma 4.7.

$$\lim_{t \rightarrow 0} G_{(\mu)}(x, t) = \delta(x).$$

Proof. Fix $x \neq 0$, for each $t \in (0, \infty)$,

$$\left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{zt - \Phi^{1/2}(z)|x}| \leq e^{\gamma t} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{-\Phi^{1/2}(z)|x}|.$$

The proof of Lemma 4.5 shows that

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| \cdot |e^{-\Phi^{1/2}(z)|x}| < \infty,$$

then by dominated convergence theorem, we can deduce that

$$\begin{aligned} \lim_{t \rightarrow 0} G_{(\mu)}(x, t) &= \lim_{t \rightarrow 0} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt - \Phi^{1/2}(z)|x}| \, dz \\ (4.3) \quad &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} \lim_{t \rightarrow 0} e^{zt - \Phi^{1/2}(z)|x}| \, dz \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x}| \, dz, \end{aligned}$$

for each $x \neq 0$. Let $z = \gamma + mi$, we have

$$(4.4) \quad \lim_{t \rightarrow 0} G_{(\mu)}(x, t) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(\gamma + mi)}{\gamma + mi} e^{-\Phi^{1/2}(\gamma + mi)|x|} \, dm.$$

Recalling the definition of the closed contour C_- and the proof of Lemma 4.4, we see the function $\frac{\Phi^{1/2}(\gamma + mi)}{\gamma + mi} e^{-\Phi^{1/2}(\gamma + mi)|x|}$ is analytic in C_- . Then

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(\gamma + mi)}{\gamma + mi} e^{-\Phi^{1/2}(\gamma + mi)|x|} \, dm &= \lim_{R \rightarrow \infty} \int_{C_{R-}} \frac{\Phi^{1/2}(\gamma + mi)}{\gamma + mi} e^{-\Phi^{1/2}(\gamma + mi)|x|} \, dm \\ &= \lim_{R \rightarrow \infty} \int_{-\pi}^0 Rie^{i\theta} \frac{\Phi^{1/2}(\gamma + Rie^{i\theta})}{\gamma + Rie^{i\theta}} e^{-\Phi^{1/2}(\gamma + Rie^{i\theta})|x|} \, d\theta, \end{aligned}$$

where $m = Re^{i\theta}$. Since $\operatorname{Re}(\gamma + Rie^{i\theta}) = \gamma - R \sin \theta \geq 0$, following from the proofs of Lemmas 4.2 and 4.3, we can deduce that

$$\begin{aligned} \operatorname{Re}(\Phi^{1/2}(\gamma + Rie^{i\theta})) &\geq \frac{\sqrt{2}}{2} |\Phi^{1/2}(\gamma + Rie^{i\theta})| \\ &\geq C_{\mu, \beta} \frac{|\gamma + Rie^{i\theta}|^\beta - |\gamma + Rie^{i\theta}|^{\beta_0}}{\ln |\gamma + Rie^{i\theta}|} \geq C \frac{R^\beta - R^{\beta_0}}{\ln R}, \end{aligned}$$

and

$$|\Phi^{1/2}(\gamma + Rie^{i\theta})| \leq C \frac{|\gamma + Rie^{i\theta}| - 1}{\ln |\gamma + Rie^{i\theta}|} \leq C \frac{|R| - 1}{\ln |R|}$$

for large R . Hence, as $R \rightarrow \infty$,

$$\begin{aligned} \left| Rie^{i\theta} \frac{\Phi^{1/2}(\gamma + Rie^{i\theta})}{\gamma + Rie^{i\theta}} e^{-\Phi^{1/2}(\gamma + Rie^{i\theta})|x|} \right| &\leq \left| \frac{Rie^{i\theta}}{\gamma + Rie^{i\theta}} \right| \cdot |\Phi^{1/2}(\gamma + Rie^{i\theta})| \cdot |e^{-\Phi^{1/2}(\gamma + Rie^{i\theta})|x|}| \\ &\leq C \frac{|R| - 1}{\ln |R|} \cdot e^{-C \frac{R^\beta - R^{\beta_0}}{\ln R} |x|} \rightarrow 0, \end{aligned}$$

which implies

$$\left| \int_{-\infty}^{+\infty} \frac{\Phi^{1/2}(\gamma + mi)}{\gamma + mi} e^{-\Phi^{1/2}(\gamma + mi)|x|} dm \right| \leq \pi \cdot C \frac{|R| - 1}{\ln |R|} \cdot e^{-C \frac{R^\beta - R^{\beta_0}}{\ln R} |x|} \rightarrow 0.$$

The above result and (4.4) show that

$$(4.5) \quad \lim_{t \rightarrow 0} G_{(\mu)}(x, t) = 0 \text{ for } x \neq 0.$$

Now, we are in the position to calculate $\int_{-\infty}^{\infty} \lim_{t \rightarrow 0} G_{(\mu)}(x, t) dx$. Equation (4.3) gives

$$\begin{aligned} \int_{-\infty}^{\infty} \lim_{t \rightarrow 0} G_{(\mu)}(x, t) dx &= \int_{-\infty}^0 \lim_{t \rightarrow 0} G_{(\mu)}(x, t) dx + \int_0^{\infty} \lim_{t \rightarrow 0} G_{(\mu)}(x, t) dx \\ &= \int_{-\infty}^0 \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x|} dz dx \\ &\quad + \int_0^{\infty} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)|x|} dz dx \\ &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_{-\infty}^0 \frac{\Phi^{1/2}(z)}{2z} e^{\Phi^{1/2}(z)x} dx dz \\ &\quad + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_0^{\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)x} dx dz. \end{aligned}$$

Now Lemma 4.2 and the fact that $\operatorname{Re} z = \gamma > 0$ shows that

$$\begin{aligned} \int_{-\infty}^0 \frac{\Phi^{1/2}(z)}{2z} e^{\Phi^{1/2}(z)x} dx &= \frac{e^{\Phi^{1/2}(z)x}}{2z} \Big|_{-\infty}^0 = \frac{1}{2z}, \\ \int_0^{\infty} \frac{\Phi^{1/2}(z)}{2z} e^{-\Phi^{1/2}(z)x} dx &= \frac{e^{-\Phi^{1/2}(z)x}}{2z} \Big|_{\infty}^0 = \frac{1}{2z}. \end{aligned}$$

Therefore, $\int_{-\infty}^{\infty} \lim_{t \rightarrow 0} G_{(\mu)}(x, t) dx = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{1}{2z} \cdot 2 dz = 1$, which together with (4.5) yields the conclusion. \square

Lemma 4.7 allows us to make the definition

$$(4.6) \quad G_{(\mu)}(x, 0) = \lim_{t \rightarrow 0} G_{(\mu)}(x, t) = \delta(x).$$

4.2. The Theta functions: $\theta_{(\mu)}(x, t)$ and $\bar{\theta}_{(\mu)}(x, t)$. One very useful way to represent solutions to initial value problems for a parabolic equation is through the θ -function, [2]. For the case of the heat equation if we let $K(x, t)$ denote the fundamental solution, then set $\theta(x, t) = \sum_{m=-\infty}^{\infty} K(x + 2m, t)$. The value of this function lies in the following result. If $u_t - u_{xx} = 0$, $u(0, t) = f_0(t)$, $u(1, t) = f_1(t)$, $u(x, 0) = u_0(x)$, then $u(x, t)$ has the representation

$$(4.7) \quad \begin{aligned} u(x, t) = & \int_0^1 [\theta(x - \xi, t) - \theta(x + \xi, t)] u_0(\xi) d\xi \\ & - 2 \int_0^t \frac{\partial \theta}{\partial x}(x, t - \tau) f_0(\tau) d\tau + 2 \int_0^t \frac{\partial \theta}{\partial x}(x - 1, t - \tau) f_1(\tau) d\tau. \end{aligned}$$

A generalization to the case of the fractional equation $D_t^\alpha - u_{xx} = 0$ for a fixed α , $0 < \alpha \leq 1$ can be found in [21]. Our aim is to extend this representation result to the distributed fractional order case.

Definition 4.1. We define for each $\mu(\alpha)$ which satisfies Assumption 2.1,

$$\theta_{(\mu)}(x, t) = \sum_{m=-\infty}^{\infty} G_{(\mu)}(x + 2m, t).$$

The uniform convergence and smoothness property of $\theta_{(\mu)}(x, t)$ are established by the next lemma.

Lemma 4.8. $\theta_{(\mu)}(x, t)$ is an even function on x and uniformly convergent on $(0, 2) \times (0, T)$ for any positive T . Then $\theta_{(\mu)}(x, t) \in C^\infty((0, 2) \times (0, \infty))$.

Proof. The even symmetric property follows from the definitions of $G_{(\mu)}(x, t)$ and $\theta_{(\mu)}(x, t)$ directly.

Given a positive T , fix $(x, t) \in (0, 2) \times (0, T)$, by Lemma 4.2 we have

$$(4.8) \quad \begin{aligned} \sum_{|m| > N} |G_{(\mu)}(x + 2m, t)| & \leq \left| \frac{1}{2\pi i} \sum_{|m| > N} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} e^{zt - \Phi^{1/2}(z)|x + 2m|} dz \right| \\ & = \left| \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi^{1/2}(z)}{2z} \sum_{|m| > N} e^{zt - \Phi^{1/2}(z)|x + 2m|} dz \right| \\ & \leq \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| e^{\gamma t} \sum_{|m| > N} e^{-\operatorname{Re}(\Phi^{1/2}(z))|x + 2m|} dz \\ & \leq \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| e^{\gamma t} \sum_{|m| > N} e^{-\frac{\sqrt{\gamma}}{2} |\Phi^{1/2}(z)||x + 2m|} dz. \end{aligned}$$

For the series $\sum_{|m|>N} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)||x+2m|}$, Lemma 4.3 shows that

$$\begin{aligned}
& \sum_{|m|>N} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)||x+2m|} \\
&= (1 - e^{-\sqrt{2}|\Phi^{1/2}(z)|})^{-1} (e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N+2+x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N+2-x)}) \\
&= \frac{e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(2N-2)}}{1 - e^{-\sqrt{2}|\Phi^{1/2}(z)|}} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|} (e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(3+x)} + e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|(3-x)}) \\
&\leq 2(1 - e^{-\sqrt{2}(C_{\mu,\beta} \frac{\gamma^{\beta-\gamma\beta_0}}{\ln \gamma})^{1/2}})^{-1} (e^{-\frac{\sqrt{2}}{2}(C_{\mu,\beta} \frac{\gamma^{\beta-\gamma\beta_0}}{\ln \gamma})^{1/2}})^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|} \\
&\leq A_\gamma C_\gamma^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|}
\end{aligned}$$

where

$$A_\gamma = 2(1 - e^{-\sqrt{2}(C_{\mu,\beta} \frac{\gamma^{\beta-\gamma\beta_0}}{\ln \gamma})^{1/2}})^{-1}, \quad 0 < C_\gamma = e^{-\frac{\sqrt{2}}{2}(C_{\mu,\beta} \frac{\gamma^{\beta-\gamma\beta_0}}{\ln \gamma})^{1/2}} < 1$$

only depend on $\gamma > 0$. Inserting the above result into (4.8) yields

$$\sum_{|m|>N} |G_{(\mu)}(x+2m, t)| \leq \frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| e^{\gamma t} A_\gamma C_\gamma^{2N-2} e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|} dz.$$

Meanwhile, from the proof of Lemma 4.5, we have

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Phi^{1/2}(z)}{2z} \right| e^{-\frac{\sqrt{2}}{2}|\Phi^{1/2}(z)|} dz < \infty.$$

Therefore,

$$\sum_{|m|>N} |G_{(\mu)}(x+2m, t)| \leq C C_\gamma^{2N-2}$$

where the constant C only depends on T , γ and $0 < C_\gamma < 1$ only depends on γ . We conclude from this that for each $\epsilon > 0$, \exists sufficiently large $N \in \mathbb{N}$ independent of x, t such that

$$\sum_{|m|>N} |G_{(\mu)}(x+2m, t)| < \epsilon \text{ for each } (x, t) \in (0, 2) \times (0, T),$$

which implies the uniform convergence of the series. Then the smoothness results follow from Lemma 4.6 and the uniform convergence. \square

Now we introduce the definition of $\bar{\theta}_{(\mu)}(x, t)$ and state some of its properties.

Definition 4.2.

$$\bar{\theta}_{(\mu)}(x, t) = \left(I^{(\mu)} \frac{\partial^2 \theta_{(\mu)}}{\partial t \partial x} \right) (x, t), \quad (x, t) \in (0, 2) \times (0, \infty).$$

Lemma 4.9. $D^{(\mu)} \theta_{(\mu)}(x, t) = (\theta_{(\mu)}(x, t))_{xx}$, $D^{(\mu)} \bar{\theta}_{(\mu)}(x, t) = (\bar{\theta}_{(\mu)}(x, t))_{xx}$.

Proof. The first equality follows from the fact $D^{(\mu)} G_{(\mu)}(x, t) = (G_{(\mu)}(x, t))_{xx}$ and the uniform convergence of the series representation.

For the second equality, Lemma 2.1 yields $D^{(\mu)}\bar{\theta}_{(\mu)} = D^{(\mu)}I^{(\mu)}\frac{\partial^2\theta_{(\mu)}}{\partial t\partial x} = \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}$ and this together with the first equality and Lemma 4.8 then gives

$$\begin{aligned} (\bar{\theta}_{(\mu)})_{xx} &= I^{(\mu)}\frac{\partial^2}{\partial t\partial x}\left(\frac{\partial^2\theta_{(\mu)}}{\partial x^2}\right) = I^{(\mu)}\frac{\partial^2}{\partial t\partial x}D^{(\mu)}\theta_{(\mu)} = I^{(\mu)}\frac{\partial}{\partial t}D^{(\mu)}\left(\frac{\partial\theta_{(\mu)}}{\partial x}\right) \\ &= \kappa * \frac{\partial}{\partial t}\left[\eta * \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}\right] = \kappa * \eta * \frac{\partial^3\theta_{(\mu)}}{\partial t^2\partial x} + \kappa * \eta \cdot \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x, 0) \\ &= \int_0^t \frac{\partial^3\theta_{(\mu)}}{\partial t^2\partial x} dt + \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x, 0) \\ &= \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x, t) - \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x, 0) + \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}(x, 0) = \frac{\partial^2\theta_{(\mu)}}{\partial t\partial x}, \end{aligned}$$

which shows that the second equality holds. \square

Lemma 4.10. *For each $\psi(t) \in L^2(0, \infty)$, we have*

$$\begin{aligned} \int_0^t \bar{\theta}_{(\mu)}(0+, t-s)\psi(s)ds &= -\frac{1}{2}\psi(t), \quad \int_0^t \bar{\theta}_{(\mu)}(1-, t-s)\psi(s)ds = 0, \\ \int_0^t \bar{\theta}_{(\mu)}(0-, t-s)\psi(s)ds &= \frac{1}{2}\psi(t), \quad \int_0^t \bar{\theta}_{(\mu)}(-1+, t-s)\psi(s)ds = 0, \quad t \in (0, \infty). \end{aligned}$$

Proof. Fix $(x, t) \in (0, 1) \times (0, \infty)$, then computing the Laplace transform yields (4.9)

$$\begin{aligned} \mathbb{L}(\bar{\theta}_{(\mu)}(x, t)) &= \mathbb{L}\left[\kappa * \left(\frac{\partial^2}{\partial t\partial x} \sum_{m=-\infty}^{+\infty} G_{(\mu)}(x, t)\right)\right] \\ &= \mathbb{L}\left[\kappa * \left(\sum_{m=-1}^{-\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt+\Phi^{1/2}(z)(x+2m)} dz \right. \right. \\ &\quad \left. \left. - \sum_{m=0}^{+\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt-\Phi^{1/2}(z)(x+2m)} dz\right)\right] \\ &= \mathbb{L}(\kappa) \cdot \mathbb{L}\left(\sum_{m=-1}^{-\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt+\Phi^{1/2}(z)(x+2m)} dz \right. \\ &\quad \left. - \sum_{m=0}^{+\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt-\Phi^{1/2}(z)(x+2m)} dz\right) \\ &= \frac{1}{\Phi(z)} \left(\sum_{m=-1}^{-\infty} \frac{\Phi(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} - \sum_{m=0}^{+\infty} \frac{\Phi(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)}\right) \\ &= \frac{e^{(x-2)\Phi^{1/2}(z)} - e^{-x\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})}, \end{aligned}$$

where the last equality follows from the fact $\text{Re}(\Phi^{1/2}(z)) > 0$ which is in turn ensured by Lemma 4.2. Therefore,

$$\begin{aligned} \mathbb{L}\left(\int_0^t \bar{\theta}_{(\mu)}(0+, t-s)\psi(s)ds\right) &= \mathbb{L}(\bar{\theta}_{(\mu)}(0+, t))\mathbb{L}(\psi(t)) = -\frac{1}{2}\mathbb{L}(\psi(t)); \\ \mathbb{L}\left(\int_0^t \bar{\theta}_{(\mu)}(1-, t-s)\psi(s)ds\right) &= \mathbb{L}(\bar{\theta}_{(\mu)}(1-, t))\mathbb{L}(\psi(t)) = 0. \end{aligned}$$

For $(x, t) \in (-1, 0) \times (0, \infty)$, we have

$$\begin{aligned} \mathbb{L}(\bar{\theta}_{(\mu)}(x, t)) &= \mathbb{L}\left[\kappa * \left(\sum_{m=0}^{-\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt+\Phi^{1/2}(z)(x+2m)} dz \right. \right. \\ &\quad \left. \left. - \sum_{m=1}^{+\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(z)}{2} e^{zt-\Phi^{1/2}(z)(x+2m)} dz \right)\right] \\ &= \frac{1}{\Phi(z)} \left(\sum_{m=0}^{-\infty} \frac{\Phi(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} - \sum_{m=1}^{+\infty} \frac{\Phi(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)} \right) \\ &= \frac{e^{x\Phi^{1/2}(z)} - e^{-(x+2)\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})}, \end{aligned}$$

which gives $\mathbb{L}(\bar{\theta}_{(\mu)}(0-, t)) = \frac{1}{2}$ and $\mathbb{L}(\bar{\theta}_{(\mu)}(-1+, t)) = 0$, and completes the proof. \square

4.3. Representation of the solution to the initial-boundary value problem. We will build the representation of the solution in this subsection from four representations in terms of the theta functions; the initial condition, the values of u at each boundary $x = 0$, $x = 1$, and the nonhomogeneous term f .

Definition 4.3.

$$\begin{aligned} u_1(x, t) &= \int_0^1 (\theta_{(\mu)}(x-y, t) - \theta_{(\mu)}(x+y, t)) u_0(y) dy; \\ u_2(x, t) &= -2 \int_0^t \bar{\theta}_{(\mu)}(x, t-s) g_0(s) ds; \\ u_3(x, t) &= 2 \int_0^t \bar{\theta}_{(\mu)}(x-1, t-s) g_1(s) ds; \\ u_4(x, t) &= \int_0^1 \int_0^t [\theta_{(\mu)}(x-y, t-s) - \theta_{(\mu)}(x+y, t-s)] \cdot \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, s) \right] ds dy. \end{aligned}$$

The following four lemmas give some properties of u_j , $j = 1, 2, 3, 4$.

Lemma 4.11. $D^{(\mu)} u_j = \frac{\partial^2 u_j}{\partial x^2}$, $j = 1, 2, 3$, $D^{(\mu)} u_4 = \frac{\partial^2 u_4}{\partial x^2} + f(x, t)$, where $(x, t) \in (0, 1) \times (0, \infty)$.

Proof. For u_1 , by Lemma 4.9, we have

$$\begin{aligned} D^{(\mu)} u_1 &= \int_0^1 (D^{(\mu)} \theta_{(\mu)}(x-y, t) - D^{(\mu)} \theta_{(\mu)}(x+y, t)) u_0(y) dy \\ &= \int_0^x (D^{(\mu)} \theta_{(\mu)}(x-y, t) - D^{(\mu)} \theta_{(\mu)}(x+y, t)) u_0(y) dy \\ &\quad + \int_x^1 (D^{(\mu)} \theta_{(\mu)}(x-y, t) - D^{(\mu)} \theta_{(\mu)}(x+y, t)) u_0(y) dy \\ &= \int_0^x [\theta_{(\mu)}(x-y, t) - \theta_{(\mu)}(x+y, t)]_{xx} u_0(y) dy \\ &\quad + \int_x^1 [\theta_{(\mu)}(x-y, t) - \theta_{(\mu)}(x+y, t)]_{xx} u_0(y) dy \\ &= \int_0^1 [\theta_{(\mu)}(x-y, t) - \theta_{(\mu)}(x+y, t)]_{xx} u_0(y) dy = \frac{\partial^2 u_1}{\partial x^2}. \end{aligned}$$

For u_2 ,

$$\begin{aligned} D^{(\mu)}u_2 &= \eta * \frac{\partial u_2}{\partial t} = -2\eta * \frac{\partial}{\partial t}(\bar{\theta}_{(\mu)} * g_0) = -2\eta * \left(\frac{\partial}{\partial t}\bar{\theta}_{(\mu)}\right) * g_0 - 2(\eta * g_0) \cdot \bar{\theta}_{(\mu)}(x, 0) \\ &= -2D^{(\mu)}\bar{\theta}_{(\mu)} * g_0 = -2(\bar{\theta}_{(\mu)})_{xx} * g_0 = (-2\bar{\theta}_{(\mu)} * g_0)_{xx} = (u_2)_{xx}. \end{aligned}$$

In an analogous fashion to the above argument, we deduce that $D^{(\mu)}u_3 = (u_3)_{xx}$.

For u_4 , using Lemmas 4.7, 2.1 and 4.8 we obtain

$$\begin{aligned} D^{(\mu)}u_4 &= \eta * \frac{\partial u_4}{\partial t} = \eta * \frac{\partial}{\partial t} \left(\int_0^1 [\theta_{(\mu)}(x-y, \cdot) - \theta_{(\mu)}(x+y, \cdot)] * \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, \cdot) \right] dy \right) \\ &= \eta * \left(\int_0^1 \frac{\partial}{\partial t} [\theta_{(\mu)}(x-y, \cdot) - \theta_{(\mu)}(x+y, \cdot)] * \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, \cdot) \right] dy \right) \\ &\quad + \eta * \left(\int_0^1 [\theta_{(\mu)}(x-y, 0) - \theta_{(\mu)}(x+y, 0)] \cdot \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, t) \right] dy \right) \\ &= \int_0^1 \eta * \frac{\partial}{\partial t} [\theta_{(\mu)}(x-y, \cdot) - \theta_{(\mu)}(x+y, \cdot)] * \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, \cdot) \right] dy \\ &\quad + \eta * \left(\int_0^1 [\delta(x-y) - \delta(x+y)] \cdot \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, t) \right] dy \right) \\ &= \int_0^1 D^{(\mu)}[\theta_{(\mu)}(x-y, \cdot) - \theta_{(\mu)}(x+y, \cdot)] * \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, \cdot) \right] dy + \eta * \frac{\partial}{\partial t} I^{(\mu)} f(x, t) \\ &= \int_0^1 [\theta_{(\mu)}(x-y, \cdot) - \theta_{(\mu)}(x+y, \cdot)]_{xx} * \left[\frac{\partial}{\partial t} I^{(\mu)} f(y, \cdot) \right] dy + D^{(\mu)} I^{(\mu)} f(x, t) \\ &= (u_4)_{xx} + f(x, t). \end{aligned}$$

□

Lemma 4.12. $\lim_{t \rightarrow 0} u_1(x, t) = u_0(x)$, $\lim_{t \rightarrow 0} u_j(x, t) = 0$ for $j = 2, 3, 4$, $x \in (0, 1)$.

Proof. For each $x \in (0, 1)$, Lemmas 4.8 and 4.6 yield that

$$\begin{aligned} \lim_{t \rightarrow 0} u_1 &= \int_0^1 (\theta_{(\mu)}(x-y, 0) - \theta_{(\mu)}(x+y, 0)) u_0(y) dy \\ &= \int_0^1 \sum_{m=-\infty}^{\infty} (\delta(x-y+2m) - \delta(x+y+2m)) u_0(y) dy = \int_0^1 \delta(x-y) u_0(y) dy = u_0(x). \end{aligned}$$

The other result follows directly from the definitions of u_2 , u_3 and u_4 . □

Lemma 4.13. $u_j(0, t) = u_j(1, t) = 0$, for $j = 1, 4$ and $t \in (0, \infty)$.

Proof. Since $\theta_{(\mu)}(x, t)$ is even on x which is stated in Lemma 4.8, then

$$u_1(0, t) = \int_0^1 (\theta_{(\mu)}(0-y, t) - \theta_{(\mu)}(0+y, t)) u_0(y) dy = 0.$$

We also have

$$\begin{aligned} u_1(1, t) &= \int_0^1 (\theta_{(\mu)}(1-y, t) - \theta_{(\mu)}(1+y, t)) u_0(y) dy \\ &= \int_0^1 (\theta_{(\mu)}(y-1, t) - \theta_{(\mu)}(1+y, t)) u_0(y) dy \\ &= \int_0^1 \left[\sum_{m=-\infty}^{\infty} G_{(\mu)}(y-1+2m, t) - \sum_{m=-\infty}^{\infty} G_{(\mu)}(y+1+2m, t) \right] u_0(y) dy \\ &= \int_0^1 \left[\sum_{q=-\infty}^{\infty} G_{(\mu)}(y+1+2q, t) - \sum_{m=-\infty}^{\infty} G_{(\mu)}(y+1+2m, t) \right] u_0(y) dy = 0, \end{aligned}$$

where $q = m - 1$.

Following from the above proof, we obtain the conclusion for u_4 . \square

Lemma 4.14. $u_2(0, t) = g_0(t)$, $u_2(1, t) = 0$, $u_3(0, t) = 0$, $u_3(1, t) = g_1(t)$, for $t \in (0, \infty)$.

Proof. The proof follows from Lemma 4.10 directly. \square

Now we can state

Theorem 4.4 (Representation theorem). *There exists a unique solution $u(x, t)$ of Equations (4.1), which has the representation $u(x, t) = \sum_{j=1}^4 u_j$.*

Proof. The existence follows from Lemmas 4.11, 4.12, 4.13 and 4.14; while the uniqueness is ensured by Corollary 3.1. \square

5. DETERMINING THE DISTRIBUTED COEFFICIENT $\mu(\alpha)$

In this section we state and prove two uniqueness theorems for the recovery of the distributed derivative μ . We show that by measuring the solution along a time trace from a fixed location x_0 one can use this data to uniquely recover $\mu(\alpha)$. This time trace can be one where the sampling point is located within the interior of $\Omega = (0, 1)$ and we measure $u(x_0, t)$, or we measure the flux at x^* ; $u_x(x^*, t)$ where $0 < x^* \leq 1$. This latter case therefore includes measuring the flux on the right-hand boundary $x = 1$.

First we give the definition of the admissible set Ψ according to Assumption 2.1.

Definition 5.1. *Define the set Ψ by*

$$\Psi := \{\mu \in C^1[0, 1] : \mu \geq 0, \mu(1) \neq 0, \mu(\alpha) \geq C_\Psi > 0 \text{ on } (\beta_0, \beta_1)\},$$

where the constant $C_\Psi > 0$ and the interval $(\beta_0, \beta_1) \subset (0, 1)$ only depend on Ψ .

We introduce the functions $F(y; x_0)$ and $F_f(y; x^*)$ in the next two lemmas.

Lemma 5.1. *Define the function $F(y; x_0) \in C^1((0, \infty), \mathbb{R})$ as*

$$F(y; x_0) = \frac{e^{(x_0-2)y} - e^{-x_0y}}{2(1 - e^{-2y})},$$

where $x_0 \in (0, 1)$ is a constant. Then the function $F(y; x_0)$ is strictly increasing on the interval $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty) \subset (0, \infty)$.

Proof. Since $x_0 \in (0, 1)$, $e^{(x_0-2)y} - e^{-x_0y} < 0$ and $2(1 - e^{-2y}) > 0$ on $(0, \infty)$. A direct calculation now yields

$$\frac{d}{dy}(e^{(x_0-2)y} - e^{-x_0y}) = (x_0 - 2)e^{(x_0-2)y} + x_0e^{-x_0y} > 0$$

for $y \in (\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. Then we have $e^{(x_0-2)y} - e^{-x_0y} < 0$ and strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. The function $2(1 - e^{-2y})$ is obviously both positive and strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$. Hence the function $F(y; x_0)$ is also strictly increasing on $(\frac{\ln(2-x_0)-\ln x_0}{2(1-x_0)}, \infty)$, which completes the proof. \square

Lemma 5.2. *For the inverse problem with flux data, define the function $F_f(y; x^*) \in C^1((0, \infty), \mathbb{R})$ as*

$$F_f(y; x^*) = \frac{ye^{(x^*-2)y} + ye^{-x^*y}}{2(1 - e^{-2y})},$$

where $x^* \in (0, 1]$ is a constant. Then the function $F_f(y; x^*)$ is strictly decreasing on the interval $(1/x^*, \infty) \subset (0, \infty)$.

Proof.

$$\begin{aligned} \frac{\partial F_f}{\partial y}(y; x^*) &= \frac{((x^* - 2)y + 1)e^{(x^* - 2)y} + (1 - x^*y)e^{-x^*y}}{2(1 - e^{-2y})^2} \\ &\quad + \frac{(-x^*y - 1)e^{(x^* - 4)y} + ((x^* - 2)y - 1)e^{(-x^* - 2)y}}{2(1 - e^{-2y})^2}, \end{aligned}$$

hence $\frac{\partial F_f}{\partial y}(y; x^*) < 0$ if $y \in (1/x^*, \infty)$ and the proof is complete. \square

For the important lemmas to follow, we need the Stone–Weierstrass and the Müntz–Szász Theorems. See the appendix for statements and references for these results.

The next result shows that the set $\{(nr)^x : n \in \mathbb{N}^+\}$ is complete in $L^2[0, 1]$ for any positive integer r . We give two proofs of this important lemma.

Lemma 5.3. *For each $r \in \mathbb{N}^+$, the vector space consisting with the set of functions $\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in the space $L^2[0, 1]$, i.e.*

$$\overline{\text{span}\{(nr)^x : n \in \mathbb{N}^+\}} = L^2[0, 1]$$

w.r.t L^2 norm. In other words, the set $\{(nr)^x : n \in \mathbb{N}^+\}$ is complete in $L^2[0, 1]$.

Proof. Clearly, $\text{span}\{(nr)^x : n \in \mathbb{N}^+\}$ satisfies all the conditions of the Stone–Weierstrass Theorem, so that the closure of $\text{span}\{(nr)^x : n \in \mathbb{N}^+\}$ w.r.t the continuous norm is either $C[0, 1]$ or $\{f \in C[0, 1] : f(x_0) = 0, x_0 \in [0, 1]\}$. The two alternatives both yield that $\text{span}\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in $C[0, 1]$ with respect to the L^2 norm, which together with the fact $C[0, 1]$ is dense in $L^2[0, 1]$ gives $\text{span}\{(nr)^x : n \in \mathbb{N}^+\}$ is dense in $L^2[0, 1]$ and completes the proof.

As a second proof, if for some $h \in C[0, 1]$, $\int_0^1 (nr)^x h(x) dx = 0$ for all $n \in \mathbb{N}^+$ then $\int_0^1 e^{x \log(rn)} h(x) dx = 0$ and with the change of variables $y = e^x$ this becomes $\int_1^e y^{\log(rn)} \tilde{h}(y) dy = 0$ for all $n \in \mathbb{N}^+$ where $\tilde{h}(y) = h(\log(y))/y$. Since $\sum_{n=1}^{\infty} 1/\log(rn)$ diverges, the Müntz–Szász theorem shows that $\tilde{h} = 0$ and hence $h(x) = 0$. \square

We now have the main result of this paper.

Theorem 5.2 (Uniqueness theorem for the inverse problem). *In the DDE (4.1), set $u_0 = g_1 = f = 0$ and let g_0 satisfy the following condition*

$$(Lg_0)(z) \neq 0 \text{ for } z \in (0, \infty).$$

Given $\mu_1, \mu_2 \in \Psi$, denote the two weak solutions with respect to μ_1 and μ_2 by $u(x, t; \mu_1)$ and $u(x, t; \mu_2)$ respectively. Then for any $x_0 \in (0, 1)$ and $x^ \in (0, 1]$, either*

$$u(x_0, t; \mu_1) = u(x_0, t; \mu_2)$$

or

$$\frac{\partial u}{\partial x}(x^*, t; \mu_1) = \frac{\partial u}{\partial x}(x^*, t; \mu_2), \quad t \in (0, \infty)$$

implies $\mu_1 = \mu_2$ on $[0, 1]$.

Proof. For the first case of $u(x_0, t; \mu_1) = u(x_0, t; \mu_2)$, fix $x_0 \in (0, 1)$, Theorem 4.4 yields for $k = 1, 2$:

$$u(x_0, t; \mu_k) = -2 \int_0^t \bar{\theta}_{(\mu_k)}(x_0, t-s) g_0(s) ds, \quad k = 1, 2$$

which implies

$$\int_0^t \bar{\theta}_{(\mu_1)}(x_0, t-s) g_0(s) ds = \int_0^t \bar{\theta}_{(\mu_2)}(x_0, t-s) g_0(s) ds.$$

Taking the Laplace transform in t on both sides of the above equality gives

$$\left(\mathbb{L}(\bar{\theta}_{(\mu_1)}(x_0, \cdot))\right)(z) \cdot (\mathbb{L}g_0)(z) = \left(\mathbb{L}(\bar{\theta}_{(\mu_2)}(x_0, \cdot))\right)(z) \cdot (\mathbb{L}g_0)(z).$$

Since $(\mathbb{L}g_0)(z) \neq 0$ on $(0, \infty)$, so that

$$\left(\mathbb{L}(\bar{\theta}_{(\mu_1)}(x_0, \cdot))\right)(z) = \left(\mathbb{L}(\bar{\theta}_{(\mu_2)}(x_0, \cdot))\right)(z), \text{ for } z \in (0, \infty).$$

This result and (4.9) then give

$$\frac{e^{(x_0-2)\Phi_1^{1/2}(z)} - e^{-x_0\Phi_1^{1/2}(z)}}{2(1 - e^{-2\Phi_1^{1/2}(z)})} = \frac{e^{(x_0-2)\Phi_2^{1/2}(z)} - e^{-x_0\Phi_2^{1/2}(z)}}{2(1 - e^{-2\Phi_2^{1/2}(z)})}, \quad z \in (0, \infty),$$

where

$$\Phi_j(z) = \int_0^1 \mu_j(\alpha) z^\alpha d\alpha, \quad j = 1, 2.$$

The definition of Ψ and the fact $z \in (0, \infty)$ yield $\Phi_j^{1/2}(z) \in (0, \infty)$ and hence we can rewrite the above equality as

$$(5.1) \quad F(\Phi_1^{1/2}(z); x_0) = F(\Phi_2^{1/2}(z); x_0), \quad z \in (0, \infty),$$

where the function F comes from Lemma 5.1.

Since $x_0 \in (0, 1)$, it is obvious that $\frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)} > 0$. Then we can pick a large $N^* \in \mathbb{N}^+$ such that

$$\int_{\beta_0}^{\beta_1} C_\Psi \cdot (N^*)^\alpha d\alpha > \left(\frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)} \right)^2,$$

which together with the definition of Ψ gives that for each $z \in (0, \infty)$ with $z \geq N^*$, $\Phi_j(z) \in (0, \infty)$ and

$$\Phi_j^{1/2}(z) > \frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)}, \quad j = 1, 2.$$

This result means that

$$(5.2) \quad \Phi_j^{1/2}(nN^*) > \frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)}, \quad j = 1, 2, \quad n \in \mathbb{N}^+.$$

Lemma 5.1 shows that $F(\cdot; x_0)$ is strictly increasing on the interval $(\frac{\ln(2-x_0) - \ln x_0}{2(1-x_0)}, \infty)$, which together with (5.1) and (5.2) yields

$$\Phi_1^{1/2}(nN^*) = \Phi_2^{1/2}(nN^*), \quad n \in \mathbb{N}^+,$$

that is $\Phi_1(nN^*) = \Phi_2(nN^*)$, $n \in \mathbb{N}^+$, sequentially, we have

$$\int_0^1 (\mu_1(\alpha) - \mu_2(\alpha))(nN^*)^\alpha d\alpha = 0, \quad n \in \mathbb{N}^+.$$

We can rewrite the above result as $\langle \mu_1(\alpha) - \mu_2(\alpha), (nN^*)^\alpha \rangle = 0$ for $n \in \mathbb{N}^+$. From the completeness of $\{(nN^*)^\alpha : n \in \mathbb{N}^+\}$ in $L^2[0, 1]$ which is ensured by Lemma 5.3, we have $\mu_1 - \mu_2 = 0$ in $L^2[0, 1]$, that is, $\|\mu_1 - \mu_2\|_{L^2[0, 1]} = 0$, which together with the continuity of μ_1 and μ_2 shows that $\mu_1 = \mu_2$ on $[0, 1]$.

For the case of $\frac{\partial u}{\partial x}(x^*, t; \mu_1) = \frac{\partial u}{\partial x}(x^*, t; \mu_2)$, following (4.9) we have

$$\begin{aligned} \mathbb{L} \left(\frac{\partial \bar{\theta}_{(\mu)}}{\partial x}(x, t) \right) &= \mathbb{L} \left[\kappa * \left(\frac{\partial^3}{\partial t \partial x^2} \sum_{m=-\infty}^{\infty} G_{(\mu)}(x, t) \right) \right] \\ &= \mathbb{L} \left[\kappa * \mathbb{L}^{-1} \left(\sum_{m=-1}^{-\infty} \frac{\Phi^{3/2}(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} dz + \sum_{m=0}^{\infty} \frac{\Phi^{3/2}(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)} \right) \right] \\ &= \frac{1}{\Phi(z)} \left(\sum_{m=-1}^{-\infty} \frac{\Phi^{3/2}(z)}{2} e^{\Phi^{1/2}(z)(x+2m)} + \sum_{m=0}^{\infty} \frac{\Phi^{3/2}(z)}{2} e^{-\Phi^{1/2}(z)(x+2m)} \right) \\ &= \frac{\Phi^{1/2}(z) e^{(x-2)\Phi^{1/2}(z)} + \Phi^{1/2}(z) e^{-x\Phi^{1/2}(z)}}{2(1 - e^{-2\Phi^{1/2}(z)})}. \end{aligned}$$

Following the proof for the case $u(x_0, t; \mu_1) = u(x_0, t; \mu_2)$, we can deduce $\mu_1 = \mu_2$ from the above result and Lemmas 5.2 and 5.3. \square

Remark 5.1. In this paper we have considered only the uniqueness question for the function $\mu(\alpha)$. Certainly, one would like to know under what conditions this function can be effectively recovered from the given data. Clearly this is an important question, but we caution there are many difficulties, especially with a mathematical analysis of the stability issue of μ in terms of the overposed data either $u(x_0, t)$ or $\frac{\partial u}{\partial x}(x^*, t)$. One can certainly employ the representation result of section 4 to obtain a nonlinear integral equation for μ but the analysis of this is unclear. An alternative approach would be restrict the function μ as in Lemma 2.1 to ensure that κ is completely monotone and hence use Bernstein's theorem to obtain an integral representation for this function. We hope to address some of these questions in subsequent work.

APPENDIX

The uniqueness proof in section 5 requires results on the density of a certain subset of functions and we give two ways to look at this through different formulations; namely the Stone-Weierstrass and Müntz-Szász theorems. We give the statements of these results below.

The Stone-Weierstrass theorem is a generalization of Weierstrass' result of 1885 that the polynomials are dense in $C[0, 1]$ and was proved by Stone some 50 years later, [24]. If X is a compact Hausdorff space and $C(X)$ those real-valued continuous functions on X , with the topology of uniform convergence, then the question is when is a subalgebra $A(X)$ dense? A crucial notion is that of separation of points; a set A of functions defined on X is said to *separate points* if, for every $x, y \in X$, $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$. Then we have

Theorem 5.3. (*Stone-Weierstrass*). *Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X)$ which contains a non-zero constant function. Then A is dense in $C(X)$ if and only if it separates points.*

The proof can be found in standard references, for example, [6, Theorem 4.45].

The Müntz-Szász theorem, (1914-1916) is also a generalization of the Weierstrass approximation theorem; it gives a condition under which one can “thin out” the polynomials and still maintain a dense set.

Theorem 5.4. (*Müntz-Szász*) *Let $\Lambda := \{\lambda_j\}_1^\infty$ be a sequence of real positive numbers. Then the span of $\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0, 1]$ if and only if $\sum_1^\infty \frac{1}{\lambda_j} = \infty$.*

This result can be generalized to the $L^p[0, 1]$ spaces for $1 \leq p \leq \infty$, see [1].

ACKNOWLEDGMENT

The authors were partially supported by NSF Grant DMS-1620138.

REFERENCES

- [1] P. Borwein and T. Erdélyi. The full Müntz theorem in $C[0, 1]$ and $L_1[0, 1]$. *J. London Math. Soc. (2)*, 54(1):102–110, 1996.
- [2] J. R. Cannon. *The One-Dimensional Heat Equation*. Addison-Wesley, Reading, MA, 1984.
- [3] J. Cheng, J. Nakagawa, M. Yamamoto, and T. Yamazaki. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse Problems*, 25(11):115002, 16, 2009.
- [4] M. M. Djrbashian. Differential operators of fractional order and boundary value problems in the complex domain. In *The Gohberg Anniversary Collection, Operator Theory: Advances and Applications Volume 41*, pages 153–172. 1989.
- [5] A. Einstein. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Ann. Phys.*, 322(8):549–560, 1905.
- [6] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.
- [7] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications*. Springer, Heidelberg, 2014.
- [8] Y. Hatano and N. Hatano. Dispersive transport of ions in column experiments: An explanation of long-tailed profiles. *Water Resour. Res.*, 34(5):1027–1033, 1998.
- [9] Y. Hatano, J. Nakagawa, S. Wang, and M. Yamamoto. Determination of order in fractional diffusion equation. *J. Math. Industry*, 5(A):51–57, 2013.
- [10] B. Jin and W. Rundell. A tutorial on inverse problems for anomalous diffusion processes. *Inverse Problems*, 31(3):035003, 40, 2015.
- [11] J. Klafter and I. M. Sokolov. *First Steps in Random Walks*. Oxford University Press, Oxford, 2011. From tools to applications.
- [12] A. N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340(1):252–281, 2008.
- [13] G. Li, D. Zhang, X. Jia, and M. Yamamoto. Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation. *Inverse Problems*, 29(6):065014, 36, 2013.
- [14] Z. Li, Y. Liu, and M. Yamamoto. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. *Appl. Math. Comput.*, 257:381–397, 2015.
- [15] Z. Li, Y. Luchko, and M. Yamamoto. Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem. *Computers & Mathematics with Applications*, 2016.
- [16] Z. Li and M. Yamamoto. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl. Anal.*, 94(3):570–579, 2015.
- [17] Y. Luchko. Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.*, 12(4):409–422, 2009.
- [18] F. Mainardi, A. Mura, G. Pagnini, and R. Gorenflo. Time-fractional diffusion of distributed order. *J. Vib. Control*, 14(9-10):1267–1290, 2008.
- [19] E. W. Montroll and G. H. Weiss. Random walks on lattices. II. *J. Math. Phys.*, 6(2):167–181, 1965.
- [20] M. Naber. Distributed order fractional sub-diffusion. *Fractals*, 12(1):23–32, 2004.
- [21] W. Rundell, X. Xu, and L. Zuo. The determination of an unknown boundary condition in a fractional diffusion equation. *Appl. Anal.*, 92(7):1511–1526, 2013.
- [22] H. Scher and E. W. Montroll. Anomalous transit-time dispersion in amorphous solids. *Phys. Rev. B*, 12:2455–2477, Sep 1975.
- [23] I. M. Sokolov, J. Klafter, and A. Blumen. Fractional kinetics. *Physics Today*, 55(11):48–54, 2002.
- [24] M. H. Stone. The generalized Weierstrass approximation theorem. *Math. Mag.*, 21:167–184, 237–254, 1948.
- [25] E. T. Whittaker and G. N. Watson. *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*. Fourth edition. Reprinted. Cambridge University Press, New York, 1962.